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Impact of trade credit, carbon emissions, Advertising and variable holding cost on a sustainable inventory model in a Non-Instantaneous Deterioration Framework

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Anil Kumar
Department of Mathematics
MMH College, Ghaziabad

***Monu Kumar**
Department of Mathematics,
MMH College Ghaziabad
Email:monukumar170190@gmail.com

SR Singh and Deepi D
Department of Mathematics,
Chaudhary Charan Singh University,
Meerut-250001(U.P) India

*** Corresponding author**

Abstract

In today's market, businesses face many challenges of balancing minimal cost and maximum profitability with sustainable environment while managing decomposable products that undergo quality degradation over time. This model addresses these challenges by proposing a mathematical modelling that shows realistic scenario where items keep in a preservative period before deterioration starts. In this model retailer offered trade credit as a financial structure that effect ordering decisions and cash flow also. In this model we considered uniquely demand as dependent on advertisement factor while simultaneously consider carbon emissions which addressing the sustainable supply chain. Variable holding costs are modelled as a function of stock level and time which is more realistic than constant holding cost. Using optimisation technique obtain the optimal ordering quantity, total cycle time that minimise total cost. The model is solved analytically where possible and numerically with sensitivity analysis. Our findings reveal significant interactions between financial, environmental, and operational factors, demonstrating that optimal inventory policies must balance these competing objectives. Numerical examples and comparative analyses validate the model's effectiveness and highlight its practical applicability across various industry sectors dealing with perishable goods.

Keywords: - non-instantaneous items, Variable holding cost, trade credit, carbon emissions.

1. Introduction & Literature Review

With a direct impact on profitability, customer satisfaction, and competitive advantage in the fast-paced market of today, inventory continues to be a vital component of successful business operations. Despite being fundamental, the classic EOQ model frequently fails to capture the complexity of contemporary SC contexts, where companies must concurrently manage market competitiveness, environmental laws, financial restraints, and product perishability. Organizations are under increasing pressure to create inventory strategies that not only save costs but also address sustainability issues while preserving market competitiveness as global marketplaces get more interconnected and environmentally conscientious. Managing decaying objects has special difficulties that have attracted a lot of scientific interest in recent years. Over time, the quality of many products, from pharmaceuticals and fresh fruit to fashion items and electronic components, deteriorates, potentially resulting in losses and disposal expenses.

The use of power demand and the EPQ model to inventory management is examined in this research. These models work well for goods whose demand peaks at the start of their cycles (for example, bread, yogurt, fresh fruit) or at the end (for example, sugar, oil). Although some defective items are unavoidable in production systems and require inspection, reworking, or disposal at the conclusion of each replenishment period, the EPQ model assumes that all items are produced with acceptable quality. Real-world situations frequently involve fluctuating demand patterns, in contrast to traditional models that assume constant demand rates. In order to demonstrate its practical application, the study uses the power demand pattern, which is defined by a two-parameter Weibull distribution for degradation, rather than a constant deterioration rate technique.The contributions of this study are multi fold. First, we present one of the first models to simultaneously consider non-instantaneous deterioration with trade credit, advertising, carbon emissions, and variable holding costs, providing a more realistic representation of modern inventory systems. Second, we develop solution procedures that enable practitioners to implement the model despite its complexity. Third, we offer insights into how environmental considerations affect traditional inventory-finance-marketing decisions, contributing to the growing literature on sustainable operations. Finally, our sensitivity analyses reveal critical parameter relationships that can guide managerial decision-making across various industry contexts.

The remainder of this paper is organized as follows. Section 1 reviews relevant literature across the domains of deteriorating inventory, trade credit, advertising, carbon emissions, and variable holding costs. Section 2 presents the problem description, notation, and assumptions. Section 3 develops the mathematical model and derives optimality conditions. Section 4 outlines solution procedures for determining optimal policies. Section 5 presents numerical examples and sensitivity analyses to validate the model and derive managerial insights. Section 6 discusses theoretical and practical implications of our findings. Finally, Section 7 concludes the paper with a summary of contributions and directions for future research.

Ghare and Schrader (1963) pioneered the concept of inventory deterioration, followed by researchers like Covert and Philip (1973) and Goyal and Giri (2001), who examined constant and time-dependent deterioration rates. Non-instantaneous deterioration, where items remain fresh for a certain period before beginning to deteriorate, was introduced to better reflect real product life cycles. Time-Dependent Demand has also been extensively explored. Models by Donaldson (1979) and Silver et al. (1998) considered linearly increasing demand, while more recent work includes demand influenced by price, advertising, and seasonal factors (Sana, 2007; Mandal and Phaujdar, 2001). These studies emphasize the importance of aligning inventory strategies with changing consumer behaviour over time. Advertisement Effects on Demand have been incorporated into models to reflect market-driven demand generation. Studies by Ray and Chaudhuri (1997) and Abad (2001) investigated the impact of advertisement expenditure on sales volume, indicating that promotional strategies can effectively influence inventory dynamics. Saha and Chaudhuri (2009) proposed an EOQ model where demand is a function of both price and advertisement, highlighting the interdependence of marketing and operations. Trade Credit in Inventory Systems has been analysed as a financial strategy that allows buyers to delay payments. Goyal (1985) introduced one of the earliest models incorporating trade credit, which was later extended by Chung and Huang (2003) to include variable credit periods. Teng et al. (2010) further examined the optimal replenishment policy under permissible delay in payments. These models demonstrate that trade credit can be a powerful tool to improve cash flow and inventory performance. Carbon Emission Considerations in inventory models have become increasingly important due to global environmental concerns and regulatory pressures. Researchers like Hua et al. (2011) and He et al. (2010) introduced green inventory models that integrate carbon footprint metrics into the cost function. Recent works (e.g., Sarkar et al., 2015; Taleizadeh et al., 2020) incorporate emission-dependent costs and explore trade-offs between environmental responsibility and economic performance. Variable Holding Costs have also been investigated as a response to dynamic storage environments. Shah and Pandey (2009) considered time-dependent holding costs, while Tiwari et al. (2018) proposed models where holding costs vary with stock levels and time. These formulations better reflect real-world storage scenarios where costs are influenced by electricity usage, perishability, and space limitations. Singh, *et al* (2022) developed model with the impact of preservation technology investment and different carbon emission policies. Singh ***et al*** (2022) formulated an inventory model for price-sensitive demand and preservation investment under partial backlogging. Singh, *et al* (2023, August) formulate and economic order quantity model for stock-dependent demand with setup cost dependent on population under green environment. Singh and Singh.

(2024) developed a sustainable inventory model with renewable energy under green environment. They also consider effect of circular economy.

This research aims to fill that gap by proposing a holistic model that reflects the multifaceted realities of modern inventory systems and provides actionable insights through sensitivity analysis and numerical simulations.

2. Assumptions & Notations

2.1 Assumptions

The following assumptions are used in this research

- Items are subject to non-instantaneous deterioration, meaning they remain usable for a certain period before they start to deteriorate.
- The demand rate is a function of time, possibly increasing or decreasing over the inventory cycle, and is influenced by advertising efforts. Such that $D' = a - bp + ct = D + ct$
- Carbon emission considered for holding items, purchasing items and deteriorating items.
- Preservation technology use for protect the inventory and the factor is considered as $\omega(\xi) = e^{-\xi\vartheta}$.
- Trade credit scheme is considered for this model

2.2. Notation:

The following notations are used in this research

Parameters	Descriptions
ϑ	Scaling parameter
ξ	Investment in preservation technology
a	Scaling parameter
b	Scaling parameter
c	Scaling parameter
C_{dc}	Disposal cost
θ	Deterioration rate
I_e	Interest earns
B	Carbon emission due to Placing the order
R	Carbon emission due to holding item
C_s	Shortage cost
C_l	Lost sale cost
β	Backlogging rate
O_r	Ordering cost

C_h	Holding cost
C_h'	Variable holding cost
C_d	Deterioration cost
C_p	Purchasing cost
r	Inflation rate
M	Trade credit period
I_c	Interest charge
H	Carbon emission due to disposal of item
λ	Carbon tax
t_d	Time for non-instantaneous period
Decision variables	
t_1	Time where deterioration start
T	Total cycle length
p	Selling price

3. Mathematical Model

At time t=0 an enterprise purchases Q unit of items. From starting to t_d the inventory decreases due to demand only and after t_d till t_1 inventory decreases due to deterioration and demand. And after t_1 shortage occur till time T. the inventory level graphically represented below in Fig.1.

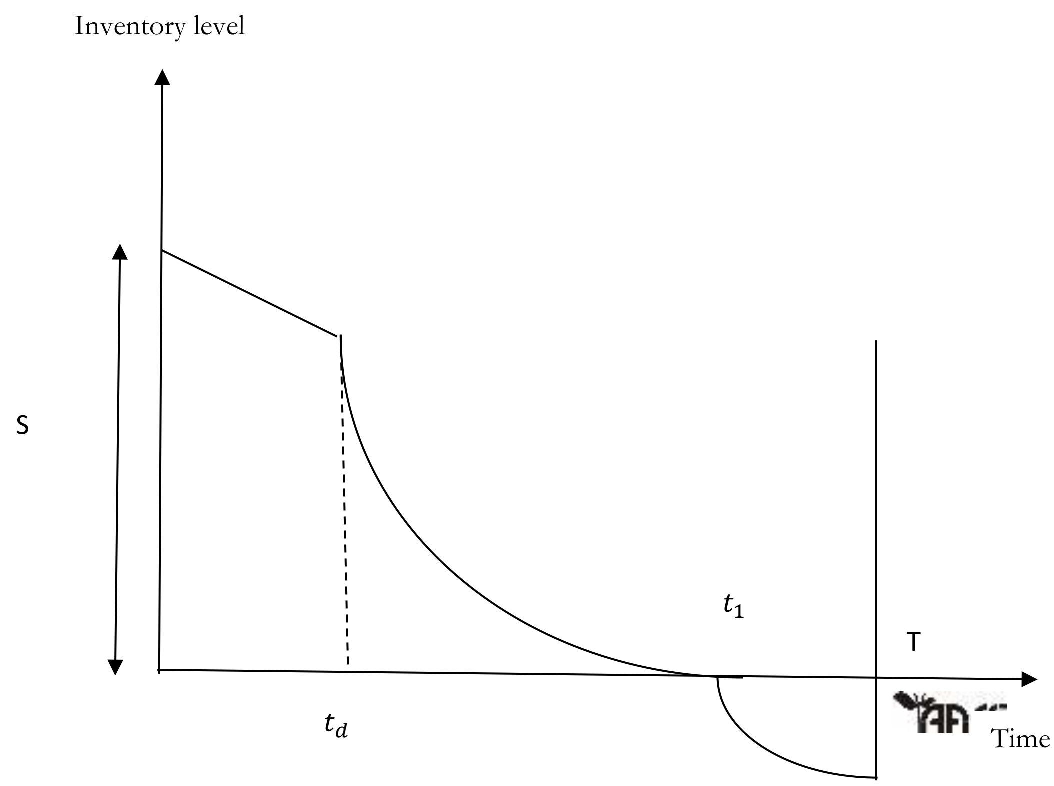


Fig 1- Inventory level with respect to time.

The differential equations of this model are:

$$\frac{dI_1(t)}{dt} = -D - ct, 0 \le t \le t_d \quad (1)$$

$$\frac{dI_2(t)}{dt} = -D - ct - \theta\omega(\xi)I_2(t), t_d \le t \le t_1 \quad (2)$$

$$\frac{dI_3(t)}{dt} = -D, t_1 \le t \le T \quad (3)$$

With boundary conditions

$$I(0) = S, I(t_1) = 0, I_3(T) = -R, I_1(t_d) = I_2(t_d) \quad (4)$$

The solution of these differential equations is:

$$I_1(t) = D(t_d - t) + \frac{c}{2}\left(t_d^2 - t^2\right) + \frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1 - t_d)} - 1\right) + \frac{c}{\theta\omega(\xi)}\left(t_1 e^{\theta\omega(\xi)(t_1 - t_d)} - t_d\right) + \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1 - e^{\theta\omega(\xi)t_d}\right)$$

(5)

$$I_2(t) = \frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1 - t)} - 1\right) + \frac{c}{\theta\omega(\xi)}\left(t_1 e^{\theta\omega(\xi)(t_1 - t)} - t\right) + \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1 - e^{\theta\omega(\xi)t_1}\right) \quad (6)$$

$$I_3(t) = D\beta(t_1 - T) \quad (7)$$

$$S = Dt_d + \frac{c}{2}t_d^2 + \frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1 - t_d)} - 1\right) + \frac{c}{\theta\omega(\xi)}\left(t_1 e^{\theta\omega(\xi)(t_1 - t_d)} - t_d\right) + \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1 - e^{\theta\omega(\xi)t_d}\right)$$

(8)

$$R = D\beta(T - t_2) \quad (9)$$

$$total\ quantity\ Q = S + R$$

$$Q = t_d + \frac{c}{2}t_d^2 + \frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1 - t_d)} - 1\right) + \frac{c}{\theta\omega(\xi)}\left(t_1 e^{\theta\omega(\xi)(t_1 - t_d)} - t_d\right) + \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1 - e^{\theta\omega(\xi)t_d}\right) + D\beta(T - t_2) \quad (10)$$

Total Cost per unit time:

Ordering cost= O_r (11)

Holding cost= $C_h\left(\int_0^{t_d} e^{-rt} I_1(t)\,dt + \int_{t_d}^{t_1} e^{-rt} I_2(t)dt\right) + C_h'\left(\int_0^{t_d} e^{-rt} t I_1(t)\,dt + \int_{t_d}^{t_1} e^{-rt} t I_2(t)dt\right)$

$$
= C_h \left[\left(Dt_d + \frac{c}{2} t_d^2 \right) t_d - \left(\frac{t_d e^{-rt_d}}{-r} - \frac{e^{-rt_d}}{r^2} + \frac{1}{r^2} \right) \left(1 + \frac{c}{2}\right) - (e^{-rt_d} - 1) \left\{ \frac{D}{\theta\omega(\xi)} \left(e^{\theta\omega(\xi)(t_1 - t_d)} - 1 \right) + \frac{c}{\theta\omega(\xi)} \left(t_1 e^{\theta\omega(\xi)(t_1 - t_d)} - t_d \right) + \frac{c}{(\theta\omega(\xi))^2} \left(1 - e^{\theta\omega(\xi) t_d} \right) \right\} \right] + C_h \left[\frac{D}{\theta\omega(\xi)} \left\{ \frac{2(e^{-rt_1} - e^{-rt_d})}{r} + \frac{1}{\theta\omega(\xi) + r} \left(e^{\theta\omega(\xi)(t_1 - t_d) - rt_d} - e^{-rt_1} \right) \right\} + \frac{c}{\theta\omega(\xi)} \left\{ \frac{t_1}{\theta\omega(\xi) + r} \left(e^{\theta\omega(\xi)(t_1 - t_d) - rt_d} - e^{-rt_1} \right) + (t_1 e^{-rt_1} - t_d e^{-rt_d}) + \frac{e^{-rt_1} - e^{-rt_d}}{r^2} \right\} - \frac{c}{(\theta\omega(\xi))^2} \left(1 - e^{\theta\omega(\xi) t_1} \right) (e^{-rt_1} - e^{-rt_d}) \right] + C_h' \left[\left(-\frac{t_d e^{-rt_d}}{r} - \frac{e^{-rt_d}}{r^2} + \frac{1}{r^2} \right) \left(Dt_d + \frac{c}{2} t_d^2 + \frac{D}{\theta\omega(\xi)} \left(e^{\theta\omega(\xi)(t_1 - t_d)} - 1 \right) + \frac{c}{\theta\omega(\xi)} \left(t_1 e^{\theta\omega(\xi)(t_1 - t_d)} - t_d \right) + \frac{c}{(\theta\omega(\xi))^2} \left(1 - e^{\theta\omega(\xi)(t_d)} \right) \right) - D \left(-\frac{t_d^2 e^{-rt_d}}{r} - \frac{2t_d e^{-rt_d}}{r^2} - \frac{2e^{-rt_d}}{r^3} + \frac{2}{r^3} \right) - \frac{c}{2} \left(-\frac{t_d^3 e^{-rt_d}}{r} - \frac{3t_d^2 e^{-rt_d}}{r^2} + \frac{6t_d e^{-rt_d}}{r^4} + 6/r^4 \right) \right] + C_h' \left[\frac{D}{\theta\omega(\xi)} \left(e^{-rt_1} \left(\frac{t_1}{r} + \frac{1}{r^2} - \frac{1}{(\theta\omega(\xi) + r)^2} - \frac{t_1}{\theta\omega(\xi) + r} \right) + e^{-rt_d} \left(-\frac{t_d}{r} - \frac{1}{r^2} \right) + e^{\theta\omega(\xi)(t_1 - t_d) - rt_d} \left(\frac{t_d}{\theta\omega(\xi) + r} + \frac{1}{(\theta\omega(\xi) + r)^2} \right) \right) + \frac{c}{\theta\omega(\xi)} \left(e^{-rt_1} \left(\frac{t_1^2}{r} + \frac{2t_1}{r^2} - \frac{2}{r^3} - \frac{t_1^2}{\theta\omega(\xi) + r} - \frac{t_1}{(\theta\omega(\xi) + r)^2} \right) + e^{-rt_d} \left(\frac{2}{r^3} - \frac{2t_d}{r^2} - \frac{t_d^2}{r} \right) + e^{\theta\omega(\xi)(t_1 - t_d) - rt_d} \left(\frac{t_1 t_d}{\theta\omega(\xi) + r} + \frac{t_1}{(\theta\omega(\xi) + r)^2} \right) \right) + \frac{c}{(\theta\omega(\xi))^2} \left(1 - e^{\theta\omega(\xi) t_1} \right) \left(e^{-rt_1} \left(-\frac{t_1}{r} - \frac{1}{r^2} \right) + e^{-rt_d} \left(\frac{t_d}{r} + \frac{1}{r^2} \right) \right) \right] \qquad (12)
$$

Purchase cost=$C_p Q$ (13)

Deterioration cost=$C_d \omega(\xi) \theta \int_{t_d}^{t_1} I_2(t) e^{-rt}\, dt$

$$
= C_d \omega(\xi) \theta \left[\frac{D}{\theta\omega(\xi)} \left(\frac{e^{-rt_1}}{-\theta\omega(\xi) - r} + \frac{e^{-rt_1}}{r} + \frac{e^{\theta\omega(\xi)(t_1 - t_d) - rt_d}}{\theta\omega(\xi) + r} - \frac{e^{-rt_d}}{r} \right) + \frac{c}{\theta\omega(\xi)} \left(\frac{t_1 e^{-rt_1}}{-\theta\omega(\xi) - r} - t_1 \frac{e^{\theta\omega(\xi)(t_1 - t_d) - rt_d}}{-\theta\omega(\xi) - r} + t_1 \frac{e^{-rt_1}}{r} - \frac{e^{-rt_1}}{r^2} - t_d \frac{e^{-rt_d}}{r} + \frac{e^{-rt_d}}{r^2} \right) - \frac{c}{(\theta\omega(\xi))^2} \left(1 - e^{\theta\omega(\xi) t_1} \right) \left(\frac{e^{-rt_1} - e^{-rt_d}}{r} \right) \right] \qquad (14)
$$

Disposable cost= $C_{dc} \theta \int_{t_d}^{t_1} I_2(t) e^{-rt}\, dt$

$$C_{dc}\theta\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}-t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right] \quad (15)$$

Preservation cost= ξT

Carbon Tax- Activities like placing the order, purchasing the items, holding the items in stock, and disposing of the items cause the emission of carbon.

1. Carbon emission due to Placing the order=B
2. Carbon emission due to disposable of the items= $H\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}-t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right]$
3. Carbon emission due to holding the items= $R\left\{\left[\left(Dt_d+\frac{c}{2}t_d^2\right)t_d-\left(\frac{t_de^{-rt_d}}{-r}-\frac{e^{-rt_d}}{r^2}+\frac{1}{r^2}\right)\left(1+\frac{c}{2}\right)-\left(e^{-rt_d}-1\right)\left\{\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_d}\right)\right\}\right]+C_h\left[\frac{D}{\theta\omega(\xi)}\left\{\frac{2\left(e^{-rt_1}-e^{-rt_d}\right)}{r}+\frac{1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)\right\}+\frac{c}{\theta\omega(\xi)}\left\{\frac{t_1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)+\left(t_1e^{-rt_1}-t_de^{-rt_d}\right)+\frac{e^{-rt_1}-e^{-rt_d}}{r^2}\right\}-\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(e^{-rt_1}-e^{-rt_d}\right)\right]\right\}$

Thus, the total amount of tax paid by vendors to the regulatory agency as a result of their investments in green technology is

$$\text{Tax}^c=\left[B+H\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}-t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right]+R\left\{\left[\left(Dt_d+\frac{c}{2}t_d^2\right)t_d-\left(\frac{t_de^{-rt_d}}{-r}-\frac{e^{-rt_d}}{r^2}+\frac{1}{r^2}\right)\left(1+\frac{c}{2}\right)-\left(e^{-rt_d}-1\right)\left\{\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_d}\right)\right\}\right]+\left[\frac{D}{\theta\omega(\xi)}\left\{\frac{2\left(e^{-rt_1}-e^{-rt_d}\right)}{r}+\right.\right.\right.$$

$$\frac{1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)\Big\}+\frac{c}{\theta\omega(\xi)}\left\{\frac{t_1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)+(t_1e^{-rt_1}-t_de^{-rt_d})+\frac{e^{-rt_1}-e^{-rt_d}}{r^2}\right\}-\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)(e^{-rt_1}-e^{-rt_d})\Big]\Big\}\Big]\left(1-\lambda(1-e^{-\sigma G_T})\right) \quad (17)$$

Shortage cost= $-C_s\int_{t_1}^{T} I_3(t)e^{-rt}dt$

$$=-C_s\left[D\beta t_2\left(\frac{e^{-rt_1}-e^{-rT}}{r}\right)-D\beta\left(e^{-rT}\left(\frac{1}{r^2}-\frac{T}{r}\right)+e^{-rt_1}\left(\frac{t_1}{r}-\frac{1}{r^2}\right)\right)\right] \quad (18)$$

Lost Sale cost=$C_l\int_{t_1}^{T} e^{-rt}(1-\beta)Ddt$

$$=C_l(1-\beta)D\left(\frac{e^{-rt_1}-e^{-rT}}{r}\right) \quad (19)$$

Considering the permissible delay period M, the inventory model has the following cases:
Case 1: $0\le M\le t_d$
Case 2: $t_d<M\le t_1$
Case 3: $t_1<M\le T$
Case 4: $T<M$

Interest earned

The interest earned from the accumulated sales, IE, is earned when the inventory level is positive [0, t_1]. The retailer deposits the sales revenue in an interest-bearing account at the rate of I_e/$/year. It is calculated based on the assumption that interest is earned by the retailer up to the period allowed to resolve the account, beyond this only, the interest is charged.

For Case 1: $0\le M\le t_d$

$$IE_1=pI_e\int_0^M Dte^{-rt}dt$$

$$=pI_eD\left[\frac{Me^{-rM}}{-r}-\frac{e^{-rM}}{r^2}+\frac{1}{r^2}\right] \quad (20)$$

$$IC_1=C_pI_c\int_M^{t_1} I(t)e^{-rt}\,dt$$

$$=C_pI_c\left[\int_M^{t_d} I_1(t)e^{-rt}\,dt+\int_{t_d}^{t_1} I_2(t)e^{-rt}\,dt\right]$$

$$=C_pI_c\Bigg[\frac{Dt_d}{-r}(e^{-rt_d}-e^{-rM})+D\left(\frac{t_de^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}-\frac{Me^{-rM}}{r}-\frac{e^{-rM}}{r^2}\right)-\frac{ct_d^2}{2r}(e^{-rt_d}-e^{-rM})-\frac{c}{2}\left(\frac{t_d^2e^{-rt_d}}{-r}-2t_d\frac{e^{-rt_d}}{r^2}-\frac{2e^{-rt_d}}{r^3}+\frac{M^2e^{-rM}}{r}+\frac{2Me^{-rM}}{r^2}+\frac{2e^{-rM}}{r^3}\right)-\frac{1}{r}\left(\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_d}\right)\right)(e^{-rt_d}-e^{-rM})+\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{t_1e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}+\right.$$

$$\left.\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{t_1e^{-rt_1}}{r}+\frac{e^{-rt_1}}{r^2}+\frac{t_1e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-t_d\frac{e^{-rt_d}}{r}-\frac{e^{-rt_d}}{r^2}\right)-$$

$$\left.\frac{1}{r}\left(\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\right)\left(e^{-rt_1}-e^{-rt_d}\right)\right] \quad (21)$$

For Case 2: $t_d < M \leq t_1$

$$IE_2 = pI_e\int_0^M Dte^{-rt}dt$$

$$= pI_eD\left[-\frac{Me^{-rM}}{r}+\frac{e^{-rM}}{r^2}-\frac{1}{r^2}\right] \quad (22)$$

$$IC_2 = C_pI_c\int_M^{t_1} I_2(t)e^{-rt}\,dt$$

$$= C_pI_c\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-M)-rM}}{\theta\omega(\xi)+r}-\frac{e^{-rM}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(-\frac{t_1e^{-rt_1}}{\theta\omega(\xi)+r}+\right.\right.$$

$$\left.\frac{t_1e^{-rt_1}}{r}+\frac{e^{-rt_1}}{r^2}+\frac{t_1e^{\theta\omega(\xi)(t_1-M)-rM}}{\theta\omega(\xi)+r}-\frac{Me^{-rM}}{r}-\frac{e^{-rM}}{r^2}\right)-\frac{\left(e^{-rt_1}-e^{-rM}\right)}{r}\left(\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-\right.\right.$$

$$\left.\left.\left.e^{\theta\omega(\xi)t_1}\right)\right)\right] \quad (23)$$

For Case 3: $t_1 < M \leq T$

$$IE_3 = pI_e\left[\int_0^{t_1} Dte^{-rt}\,dt+(M-t_1)\int_0^{t_1} De^{-rt}\,dt\right]$$

$$= pI_e\left[D\left(-\frac{t_1e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}+\frac{1}{r^2}\right)-\frac{(M-t_1)D}{r}\left(e^{-rt_1}-1\right)\right] \quad (24)$$

$$IC_3 = 0$$

For Case 4: $T < M$

$$IE_4 = pI_e\left[\int_0^{t_1} Dte^{-rt}\,dt+(M-T)\int_0^{T} De^{-rt}\,dt\right] \quad (25)$$

$$= pI_e\left[D\left(-\frac{t_1e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}+\frac{1}{r^2}\right)+\frac{(M-T)D}{r}\left(1-e^{-rT}\right)\right]$$

$$IC_4 = 0$$

Total cost:

$$TC_i(p,t_d,t_1,T)=\begin{cases}TC_1 & 0\leq M\leq t_d\\ TC_2 & t_d\leq M\leq t_1\\ TC_3 & t_1\leq M\leq T\\ TC_4 & T<M\end{cases} \quad (26)$$

Where:

For TC_1, TC_2, TC_3 and TC_4. (See Appendix)

4.Solution Methodology

The following solution methodology is use in this research-

First of all, find first derivative with respect to decision variables i.e. $\frac{dTC_i}{dt_1} = 0, \frac{dTC_i}{dT} = 0$and $\frac{dTC_i}{dp} = 0, i = 1,2,3$and 4
Find optimal values of decision variables.
Check the sigh of principal minor of Hassian Matrix such that H_{11}, H_{22} and H_{33}. Where

$$H_{11} = \frac{\partial^2 TC_i}{\partial t_1^2}, H_{22} = \begin{bmatrix} \frac{\partial^2 TC_i}{\partial t_1^2} & \frac{\partial^2 TC_i}{\partial t_1 \partial T} \\ \frac{\partial^2 TC_i}{\partial T \partial t_1} & \frac{\partial^2 TC_i}{\partial T^2} \end{bmatrix} \text{ and } H_{33} = \begin{bmatrix} \frac{\partial^2 TC_i}{\partial t_1^2} & \frac{\partial^2 TC_i}{\partial t_1 \partial T} & \frac{\partial^2 TC_i}{\partial t_1 \partial p} \\ \frac{\partial^2 TC_i}{\partial T \partial t_1} & \frac{\partial^2 TC_i}{\partial T^2} & \frac{\partial^2 TC_i}{\partial T \partial p} \\ \frac{\partial^2 TC_i}{\partial p \partial t_1} & \frac{\partial^2 TC_i}{\partial p \partial T} & \frac{\partial^2 TC_i}{\partial p^2} \end{bmatrix}$$

If $H_{11} > 0, H_{22} > 0$ and $H_{33} > 0$

4. Numerical and graphical analysis

To validate the proposed inventory model and demonstrate its practical implications, we conduct a comprehensive numerical analysis. The model parameters are assigned realistic values inspired by prior studies and industrial practices. We examine the effects of various factors such as trade credit, advertisement, deterioration rate, and carbon emissions on key inventory performance metrics. The values of parameters are written in following table-

Parameters	Values	Parameters	Values
ϑ	0.1	O_r	2
ξ	0.2	C_h	1
a	100	C_d	0.003
b	0.1	C_h'	40
c	10	C_p	32
C_{dc}	5	r	0.1
θ	0.01	M	1
I_e	0.36	I_c	0.025
B	1	H	1
R	100	λ	0.1
σ	0.1	G	100
β	50	t_d	15

The optimal solution of this model is
Time where shortage start t_1= 17.3914,
total cycle length T= 36.9421,
selling price p= 200.53

and total cost is 11258.84

Now showing the graphical representation with respect to different parameters.

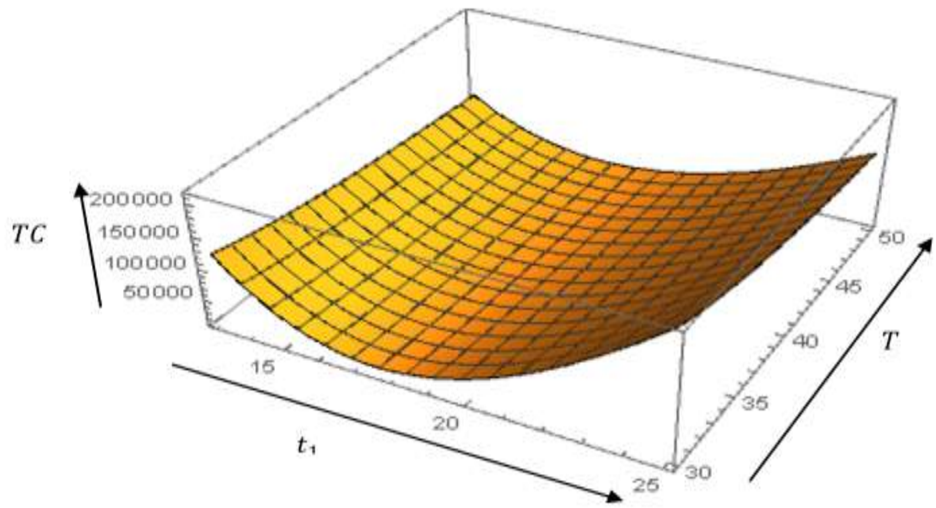
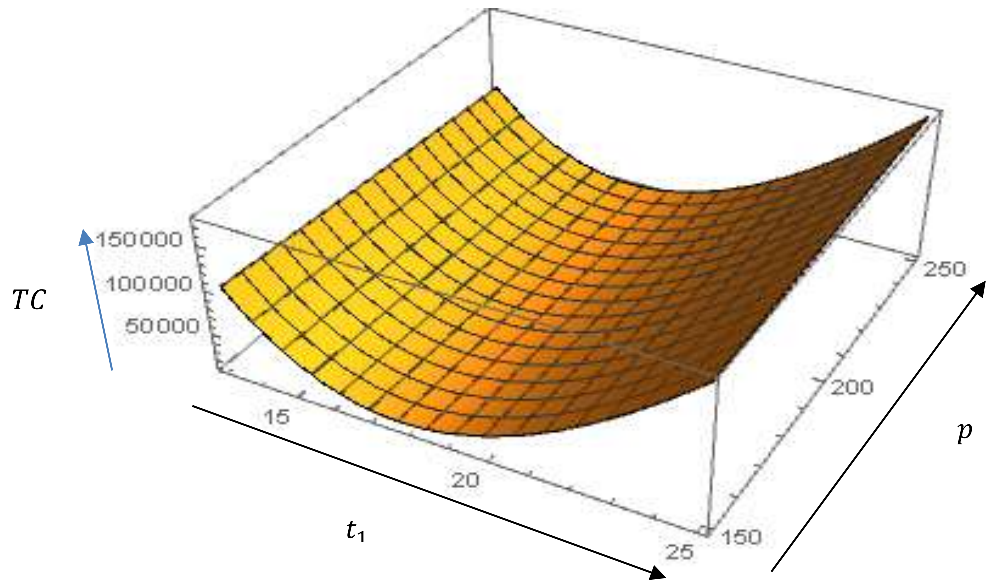


Fig 2- total cost function with respect to t_1 and T.



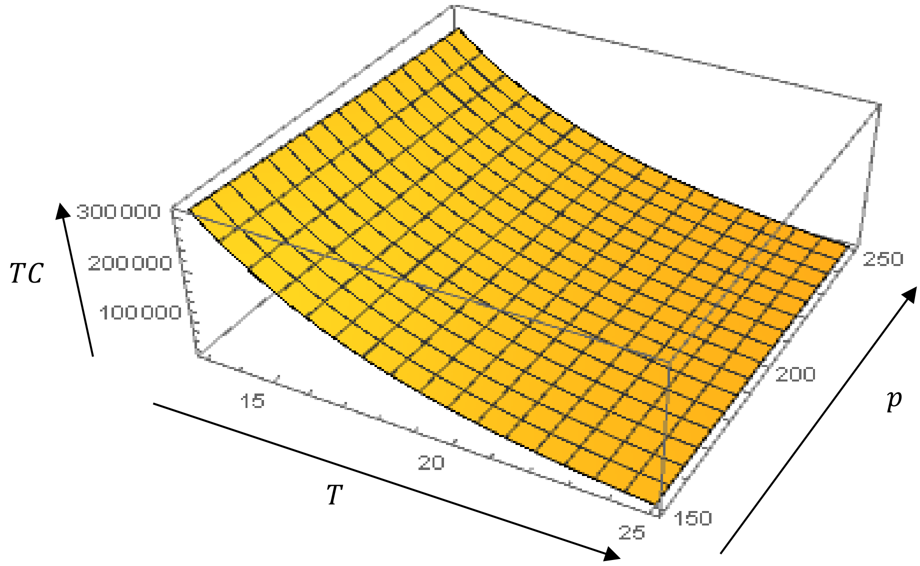


Fig 4- total cost function with respect to T and P.

6. Sensitivity Analysis :

To evaluate the robustness of the proposed inventory model and assess the impact of critical parameters on optimal decisions, a one-at-a-time sensitivity analysis is conducted. The base-case scenario is perturbed by varying one parameter at a time while keeping others constant. The effects on key performance metrics such as total inventory cost, optimal order quantity, and cycle length are recorded.

Parameters	% change	t_1	T	p	Total cost
a	+20%	15.2634	36.9421	248.37	12343.24
	+10%	16.3627	36.9421	238.36	11972.35
	0	17.3914	36.9421	200.53	11258.84
	-10%	18.2734	36.9421	192.36	10276.35
	-20%	19.2876	36.9421	182.33	10027.35
b	+20%	17.3914	37.2873	201.22	13987.24
	+10%	17.3914	36.2872	200.92	12876.35
	0	17.3914	36.9421	200.53	11258.84
	-10%	17.3914	35.2873	198.26	10265.345
	-20%	17.3914	34.2872	197.265	9268.245
	+20%	17.9762	44.2862	200.53	23768.34

c	+10%	17.6272	48.2987	200.53	20276.24
	0	17.3914	36.9421	200.53	11258.84
	-10%	17.1872	44.2872	200.53	9276.345
	-20%	17.0276	47.2873	200.53	8356.24
O_r	+20%	25.1768	30.7487	203.87	11258.84
	+10%	22.376	31.7445	205.387	11258.84
	0	17.3914	36.9421	200.53	11258.84
	-10%	15.2872	38.7437	210.27	11258.84
	-20%	11.2763	46.764	211.376	11258.84
C_h	+20%	24.2876	36.9421	304.387	25876.23
	+10%	21.2763	36.9421	176.376	21876.24
	0	17.3914	36.9421	200.53	11258.84
	-10%	15.2623	36.9421	210.376	9277.35
	-20%	12.2737	36.9421	232.736	8365.35
C'_h	+20%	17.3914	43.7622	239.272	27687.35
	+10%	17.3914	42.2786	221.26	25876.24
	0	17.3914	36.9421	200.53	11258.84
	-10%	17.3914	34.2876	197.23	36878.35
	-20%	17.3914	33.2876	186.7326	28677.23
θ	+20%	21.2768	31.276	200.53	20876.35
	+10%	19.2763	32.287	200.53	17666.35
	0	17.3914	36.9421	200.53	11258.84
	-10%	15.2763	37.4876	200.53	9765.35
	-20%	13.2862	38.2765	200.53	8265.34
r	+20%	17.3198	42.3763	102.36	11258.84
	+10%	17.3617	44.2872	167.288	11258.84
	0	17.3914	36.9421	200.53	11258.84
	-10%	17.4876	56.3873	207.365	11258.84
	-20%	17.9726	54.2762	263.2876	11258.84
C_d	+20%	38.4223	34.2876	107.27	7346.34
	+10%	29.3873	33.4763	148.26	9782.24
	0	17.3914	36.9421	200.53	11258.84
	-10%	11.2873	29.376	248.376	13876.24
	-20%	5.38781	33.276	283.287	14665.24
C_{dc}	+20%	12.3763	25.467	178.23	6868.24
	+10%	13.2876	29.3876	197.234	9762.46
	0	17.3914	36.9421	200.53	11258.84
	-10%	19.2733	39.2876	209.37	14876.24
	-20%	20.3767	44.3871	241.287	18652.456
	+20%	13.2987	11.2876	287.34	32366.24

C_p	+10%	16.3872	16.2867	249.37	26786.35
	0	17.3914	36.9421	200.53	11258.84
	-10%	22.3761	46.3876	176.23	12786.35
	-20%	26.3786	49.8762	187.263	26833.3456
I_e	+20%	11.2873	31.2876	283.27	11258.84
	+10%	14.2897	36.2762	296.35	11258.84
	0	17.3914	36.9421	200.53	11258.84
	-10%	20.3272	38.2987	256.35	11258.84
	-20%	23.3876	39.3653	136.35	11258.84
I_c	+20%	17.5927	36.9421	102.34	47677.24
	+10%	17.4267	36.9421	167.345	26844.56
	0	17.3914	36.9421	200.53	11258.84
	-10%	17.2987	36.9421	239.45	9876.24
	-20%	17.1736	36.9421	269.36	6572.245
β	+20%	17.3914	42.2872	198.24	10879.323
	+10%	17.3914	44.2762	206.24	12368.234
	0	17.3914	36.9421	200.53	11258.84
	-10%	17.3914	35.2762	108.24	18262.35
	-20%	17.3914	38.287	167.235	23693.45
C_l	+20%	29.3763	36.1276	200.53	10876.35
	+10%	26.2837	36.5622	200.53	12763.56
	0	17.3914	36.9421	200.53	11258.84
	-10%	18.2223	37.3872	200.53	17634.234
	-20%	19.2837	37.8352	200.53	19368.35

7. Observations:

The following observations are found from above table-

- When increase in parameter a then t_1 is decreasing, T is constant, p is increasing and total cost is also increasing.
- When increase in parameter b then t_1 is constant, T is increasing, p is also increasing and total cost is again increasing.
- When increase in parameter c then t_1 is increasing, T is fluctuating, p is constant and total cost is also increasing.
- When increase in parameter O_r then t_1 is increasing, T is decreasing, p is fluctuating and total cost is constant.
- When increase in parameter C_h then t_1 is increasing, T is constant, p is also increasing and total cost is again also increasing.

- When increase in parameter C_h' then t_1 is constant, T is also increasing, p is also increasing and total cost is fluctuating.
- When increase in parameter θ then t_1 is increasing, T is decreasing, p is constant and total cost is increasing.
- When increase in parameter r then t_1 is slightly decreasing, T is also decreasing, p is again decreasing and total cost is constant.
- When increase in parameter C_d then t_1 is increasing, T is fluctuating, p is decreasing and total cost is decreasing.
- When increase in parameter C_{dc} then t_1 is decreasing, T is decreasing, p is again decreasing and total cost is decreasing.
- When increase in parameter C_p then t_1 is decreasing, T is decreasing, p is fluctuating and total cost is also fluctuating.
- When increase in parameter I_e then t_1 is decreasing, T is also decreasing, p is increasing and total cost is constant.
- When increase in parameter I_c then t_1 is slightly increasing, T is constant, p is decreasing and total cost is increasing.
- When increase in parameter β then t_1 is constant, T is fluctuating, p is also fluctuating and total cost is decreasing.
- When increase in parameter C_l then t_1 is fluctuating, T is decreasing, p is constant and total cost is also decreasing.

8. Conclusion:

This research has developed a comprehensive inventory optimization model that successfully integrates non-instantaneous deterioration, trade credit financing, advertising investment, carbon emission considerations, and variable holding costs within a unified framework, addressing a significant gap in the inventory management literature. Our key findings reveal that the preservation period in non-instantaneous deterioration provides a crucial buffer for optimizing order quantities, while trade credit interactions with deterioration rates and advertising effectiveness significantly influence optimal cycle times and ordering strategies. The model demonstrates that advertising investments are particularly justified for products with shorter preservation periods, though these must be balanced against environmental impacts, as carbon pricing can lead to smaller, more frequent orders that trade off storage emissions against transportation impacts. The incorporation of variable holding costs shows that traditional constant-cost assumptions can lead to suboptimal decisions, particularly for products with extended storage periods. From a managerial perspective, the framework enables practitioners to make informed decisions that balance profitability with sustainability, providing specific guidance on coordinating inventory, financing, and marketing decisions in an integrated manner. While the model has limitations, including deterministic demand assumptions and single-item focus, it opens important avenues for future research including stochastic extensions, multi-item models, dynamic advertising strategies, and empirical validation through case studies. As businesses face increasingly complex operational environments characterized by financial constraints, environmental regulations, and intense competition, this integrated approach to inventory optimization provides valuable insights and tools for developing sustainable and profitable strategies for managing deteriorating products, ultimately contributing to more holistic supply chain management practices that serve both economic and environmental objectives.

References:

Abad, P. L. Optimal price and order size for a reseller under partial backordering. *Computers & Operations Research*, (2001),*28*(1), 53-65.

Chung, K. J., & Huang, Y. F. The optimal cycle time for EPQ inventory model under permissible delay in payments. *International Journal of Production Economics*, (2003), *84*(3), 307-318.

Ghare, P. M., & Schrader, G. H. A model for exponentially decaying inventory system. *InternationalJournal of Production Research*, (1963),*21*, 49-46.

Goyal, S. K. Economic order quantity under conditions of permissible delay in payments. *Journal of the operational research society*, (1985),335-338.

Goyal, S. K., & Giri, B. C. Recent trends in modelling of deteriorating inventory. *European Journal of operational research*, (2001),*134*(1), 1-16.

He, Y., Wang, S. Y., & Lai, K. K. An optimal production-inventory model for deteriorating items with multiple-market demand. *European Journal of Operational Research*, (2010),*203*(3), 593-600.

Hua, G., Cheng, T. C. E., & Wang, S. (2011). Managing carbon footprints in inventory management. *International journal of production economics,* (2011), *132*(2), 178-185.

Jaggi, C. K., Tiwari, S., & Gupta, M. Impact of trade credit on inventory models for Weibull distribution deteriorating items with partial backlogging in two-warehouse environment. *International Journal of Logistics Systems and Management*, (2018)*30*(4), 503-520.

Mandal, B. A., & Phaujdar, S. (1989). An inventory model for deteriorating items and stock-dependent consumption rate. *Journal of the operational Research Society*, *40*(5), 483-488.

Panda, S., Saha, S., & Basu, M. An EOQ model for perishable products with discounted selling price and stock dependent demand. *Central European Journal of Operations Research*, (2009),*17*, 31-53.

Ray, J., & Chaudhuri, K. S. An EOQ model with stock-dependent demand, shortage, inflation and time discounting. *International Journal of Production Economics*, (1997), *53*(2), 171-180.

Sana, S. S., Goyal, S. K., & Chaudhuri, K. (2007). An imperfect production process in a volume flexible inventory model. *International Journal of Production Economics*, *105*(2), 548-559.

Sarkar, B., Saren, S., & Cárdenas-Barrón, L. E. An inventory model with trade-credit policy and variable deterioration for fixed lifetime products. *Annals of Operations Research*, (2015), *229*, 677-702.

Shah, N. H., & Pandey, P. Deteriorating inventory model when demand depends on advertisement and stock display. *International Journal of Operations Research*, (2009) *6*(2), 33-44.

Silver, E. A. A simple inventory replenishment decision rule for a linear trend in demand. *Journal of the Operational Research society*, (1979), *30*(1), 71-75.

Silver, E. A., Pyke, D. F., & Peterson, R. *Inventory management and production planning and scheduling*. (1998). (Vol. 3, p. 30). New York: Wiley.

Singh, D., Singh, S. R., & Rani, M. (2022). Impact of preservation technology investment and order cost reduction on an inventory model under different carbon emission policies. In *Data analytics and artificial intelligence for inventory and supply chain management* (pp. 225-247). Singapore: Springer Nature Singapore.

Singh, S. R., & Singh, D. (2023, August). Impact of circular economy on sustainable inventory model with renewable energy under green environment. In *International conference on soft computing for problem-solving* (pp. 881-893). Singapore: Springer Nature Singapore.

Singh, S. R., Singh, D., & Rani, M. (2022, November). Inventory model for price-sensitive demand and preservation investment under partial backlogging with low carbon. In *AIP conference proceedings* (Vol. 2516, No. 1, p. 340007). AIP Publishing LLC.

Singh, S. R., Singh, D., & Rani, M. (2023, August). EPQ model for stock-dependent demand with setup cost dependent on population under green environment. In *International conference on soft computing for problem-solving* (pp. 895-906). Singapore: Springer Nature Singapore.

Taleizadeh, A. A., Hazarkhani, B., & Moon, I. Joint pricing and inventory decisions with carbon emission considerations, partial backordering and planned discounts. *Annals of Operations Research*,(2020) *290*, 95-113.

Yang, H. L., Teng, J. T., & Chern, M. S. An inventory model under inflation for deteriorating items with stock-dependent consumption rate and partial backlogging shortages. *International Journal of Production Economics*, (2010), *123*(1), 8-19.

Appendix

$$TP_1(p,t_d,t_1,T) = O_r + C_h\Bigg[\Big(Dt_d + \frac{c}{2}t_d^2\Big)t_d - \Big(\frac{t_d e^{-rt_d}}{-r} - \frac{e^{-rt_d}}{r^2} + \frac{1}{r^2}\Big)\Big(1+\frac{c}{2}\Big) -$$

$$(e^{-rt_d}-1)\Big\{\frac{D}{\theta\omega(\xi)}\big(e^{\theta\omega(\xi)(t_1-t_d)}-1\big) + \frac{c}{\theta\omega(\xi)}\big(t_1 e^{\theta\omega(\xi)(t_1-t_d)} - t_d\big) +$$

$$\frac{c}{(\theta\omega(\xi))^2}\big(1-e^{\theta\omega(\xi)t_d}\big)\Big\}\Bigg] + C_h\Bigg[\frac{D}{\theta\omega(\xi)}\Big\{\frac{2(e^{-rt_1}-e^{-rt_d})}{r} + \frac{1}{\theta\omega(\xi)+r}\big(e^{\theta\omega(\xi)(t_1-t_d)-rt_d} -$$

$$e^{-rt_1}\big)\Big\} + \frac{c}{\theta\omega(\xi)}\Big\{\frac{t_1}{\theta\omega(\xi)+r}\big(e^{\theta\omega(\xi)(t_1-t_d)-rt_d} - e^{-rt_1}\big) + (t_1e^{-rt_1} - t_d e^{-rt_d}) +$$

$$\frac{e^{-rt_1}-e^{-rt_d}}{r^2}\Big\} - \frac{c}{(\theta\omega(\xi))^2}\big(1-e^{\theta\omega(\xi)t_1}\big)(e^{-rt_1}-e^{-rt_d})\Bigg] + C_h'\Bigg[\Big(-\frac{t_d e^{-rt_d}}{r} - \frac{e^{-rt_d}}{r^2} +$$

$$\frac{1}{r^2}\Big)\Bigg(Dt_d + \frac{c}{2}t_d^2 + \frac{D}{\theta\omega(\xi)}\big(e^{\theta\omega(\xi)(t_1-t_d)}-1\big) + \frac{c}{\theta\omega(\xi)}\big(t_1 e^{\theta\omega(\xi)(t_1-t_d)} - t_d\big) +$$

$$\frac{c}{(\theta\omega(\xi))^2}\big(1-e^{\theta\omega(\xi)(t_d)}\big)\Bigg) - D\Big(-\frac{t_d^2 e^{-rt_d}}{r} - \frac{2t_d e^{-rt_d}}{r^2} - \frac{2e^{-rt_d}}{r^3} + \frac{2}{r^3}\Big) - \frac{c}{2}\Big(-\frac{t_d^3 e^{-rt_d}}{r} -$$

$$\frac{3t_d^2 e^{-rt_d}}{r^2} + \frac{6t_d e^{-rt_d}}{r^4} + 6/r^4\Big)\Bigg] + C_h'\Bigg[\frac{D}{\theta\omega(\xi)}\Bigg(e^{-rt_1}\Big(\frac{t_1}{r}+\frac{1}{r^2} - \frac{1}{(\theta\omega(\xi)+r)^2} - \frac{t_1}{\theta\omega(\xi)+r}\Big) +$$

$$e^{-rt_d}\Big(-\frac{t_d}{r}-\frac{1}{r^2}\Big) + e^{\theta\omega(\xi)(t_1-t_d)-rt_d}\Big(\frac{t_d}{\theta\omega(\xi)+r} + \frac{1}{(\theta\omega(\xi)+r)^2}\Big)\Bigg) + \frac{c}{\theta\omega(\xi)}\Bigg(e^{-rt_1}\Big(\frac{t_1^2}{r} +$$

$$\frac{2t_1}{r^2} - \frac{2}{r^3} - \frac{t_1^2}{\theta\omega(\xi)+r} - \frac{t_1}{(\theta\omega(\xi)+r)^2}\Big) + e^{-rt_d}\Big(\frac{2}{r^3} - \frac{2t_d}{r^2} - \frac{t_d^2}{r}\Big) +$$

$$e^{\theta\omega(\xi)(t_1-t_d)-rt_d}\Big(\frac{t_1t_d}{\theta\omega(\xi)+r} + \frac{t_1}{(\theta\omega(\xi)+r)^2}\Big)\Bigg) + \frac{c}{(\theta\omega(\xi))^2}\big(1-e^{\theta\omega(\xi)t_1}\big)\Bigg(e^{-rt_1}\Big(-\frac{t_1}{r} -$$

$$\frac{1}{r^2}\Big) + e^{-rt_d}\Big(\frac{t_d}{r}+\frac{1}{r^2}\Big)\Bigg)\Bigg] + C_pQ + C_d\omega(\xi)\theta\Bigg[\frac{D}{\theta\omega(\xi)}\Bigg(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r} + \frac{e^{-rt_1}}{r} +$$

$$\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r} - \frac{e^{-rt_d}}{r}\Bigg) + \frac{c}{\theta\omega(\xi)}\Bigg(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r} - t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r} + t_1\frac{e^{-rt_1}}{r} - \frac{e^{-rt_1}}{r^2} -$$

$$t_d\frac{e^{-rt_d}}{r} + \frac{e^{-rt_d}}{r^2}\Bigg) - \frac{c}{(\theta\omega(\xi))^2}\big(1-e^{\theta\omega(\xi)t_1}\big)\Big(\frac{e^{-rt_1}-e^{-rt_d}}{r}\Big)\Bigg] + C_{dc}\theta\Bigg[\frac{D}{\theta\omega(\xi)}\Bigg(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r} +$$

$$\frac{e^{-rt_1}}{r} + \frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r} - \frac{e^{-rt_d}}{r}\Bigg) + \frac{c}{\theta\omega(\xi)}\Bigg(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r} - t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r} + t_1\frac{e^{-rt_1}}{r} -$$

$$\frac{e^{-rt_1}}{r^2} - t_d\frac{e^{-rt_d}}{r} + \frac{e^{-rt_d}}{r^2}\Bigg) - \frac{c}{(\theta\omega(\xi))^2}\big(1-e^{\theta\omega(\xi)t_1}\big)\Big(\frac{e^{-rt_1}-e^{-rt_d}}{r}\Big)\Bigg] + \xi + \Big[B +$$

$$HC_{dc}\theta\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}-t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right]+R\left\{C_h\left[\left(Dt_d+\frac{c}{2}t_d^2\right)t_d-\left(\frac{t_de^{-rt_d}}{-r}-\frac{e^{-rt_d}}{r^2}+\frac{1}{r^2}\right)\left(1+\frac{c}{2}\right)-(e^{-rt_d}-1)\left\{\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_d}\right)\right\}\right]+C_h\left[\frac{D}{\theta\omega(\xi)}\left\{\frac{2\left(e^{-rt_1}-e^{-rt_d}\right)}{r}+\frac{1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)\right\}+\frac{c}{\theta\omega(\xi)}\left\{\frac{t_1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)+(t_1e^{-rt_1}-t_de^{-rt_d})+\frac{e^{-rt_1}-e^{-rt_d}}{r^2}\right\}-\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)(e^{-rt_1}-e^{-rt_d})\right]\right\}\right]\left(1-\lambda(1-e^{-\sigma G_T}\,)\right)-C_s\left[D\beta t_2\left(\frac{e^{-rt_1}-e^{-rT}}{r}\right)-D\beta\left(e^{-rT}\left(\frac{1}{r^2}-\frac{T}{r}\right)+e^{-rt_1}\left(\frac{t_1}{r}-\frac{1}{r^2}\right)\right)\right]+C_l(1-\beta)D\left(\frac{e^{-rt_1}-e^{-rT}}{r}\right)+pI_eD\left[\frac{Me^{-rM}}{-r}-\frac{e^{-rM}}{r^2}+\frac{1}{r^2}\right]-C_pI_c\left[\int_M^{t_d}I_1(t)e^{-rt}\,dt+\int_{t_d}^{t_1}I_2(t)e^{-rt}\,dt\right]$$

$$TP_2(p,t_d,t_1,T)=O_r+C_h\left[\left(Dt_d+\frac{c}{2}t_d^2\right)t_d-\left(\frac{t_de^{-rt_d}}{-r}-\frac{e^{-rt_d}}{r^2}+\frac{1}{r^2}\right)\left(1+\frac{c}{2}\right)-(e^{-rt_d}-1)\left\{\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_d}\right)\right\}\right]+C_h\left[\frac{D}{\theta\omega(\xi)}\left\{\frac{2\left(e^{-rt_1}-e^{-rt_d}\right)}{r}+\frac{1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)\right\}+\frac{c}{\theta\omega(\xi)}\left\{\frac{t_1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)+(t_1e^{-rt_1}-t_de^{-rt_d})+\frac{e^{-rt_1}-e^{-rt_d}}{r^2}\right\}-\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)(e^{-rt_1}-e^{-rt_d})\right]+C_h'\left[\left(-\frac{t_de^{-rt_d}}{r}-\frac{e^{-rt_d}}{r^2}+\frac{1}{r^2}\right)\left(Dt_d+\frac{c}{2}t_d^2+\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)(t_d)}\right)\right)-D\left(-\frac{t_d^2e^{-rt_d}}{r}-\frac{2t_de^{-rt_d}}{r^2}-\frac{2e^{-rt_d}}{r^3}+\frac{2}{r^3}\right)-\frac{c}{2}\left(-\frac{t_d^3e^{-rt_d}}{r}-\frac{3t_d^2e^{-rt_d}}{r^2}+\frac{6t_de^{-rt_d}}{r^4}+6/r^4\right)\right]+C_h'\left[\frac{D}{\theta\omega(\xi)}\left(e^{-rt_1}\left(\frac{t_1}{r}+\frac{1}{r^2}-\frac{1}{(\theta\omega(\xi)+r)^2}-\frac{t_1}{\theta\omega(\xi)+r}\right)+e^{-rt_d}\left(-\frac{t_d}{r}-\frac{1}{r^2}\right)+e^{\theta\omega(\xi)(t_1-t_d)-rt_d}\left(\frac{t_d}{\theta\omega(\xi)+r}+\frac{1}{(\theta\omega(\xi)+r)^2}\right)\right)+\frac{c}{\theta\omega(\xi)}\left(e^{-rt_1}\left(\frac{t_1^2}{r}+\frac{2t_1}{r^2}-\frac{2}{r^3}-\frac{t_1^2}{\theta\omega(\xi)+r}-\frac{t_1}{(\theta\omega(\xi)+r)^2}\right)+e^{-rt_d}\left(\frac{2}{r^3}-\frac{2t_d}{r^2}-\frac{t_d^2}{r}\right)+\right.\right.$$

$$e^{\theta\omega(\xi)(t_1-t_d)-rt_d}\left(\frac{t_1 t_d}{\theta\omega(\xi)+r}+\frac{t_1}{(\theta\omega(\xi)+r)^2}\right)\Bigg)+\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(e^{-rt_1}\left(-\frac{t_1}{r}-\frac{1}{r^2}\right)+e^{-rt_d}\left(\frac{t_d}{r}+\frac{1}{r^2}\right)\right)\Bigg]+C_pQ+C_d\omega(\xi)\theta\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}-t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right]+C_{dc}\theta\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}-t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right]+\xi+\Bigg[B+HC_{dc}\theta\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}-t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right]+R\Bigg\{C_h\left[\left(Dt_d+\frac{c}{2}t_d^2\right)t_d-\left(\frac{t_de^{-rt_d}}{-r}-\frac{e^{-rt_d}}{r^2}+\frac{1}{r^2}\right)\left(1+\frac{c}{2}\right)-\left(e^{-rt_d}-1\right)\left\{\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_d}\right)\right\}\right]+C_h\left[\frac{D}{\theta\omega(\xi)}\left\{\frac{2\left(e^{-rt_1}-e^{-rt_d}\right)}{r}+\frac{1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)\right\}+\frac{c}{\theta\omega(\xi)}\left\{\frac{t_1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)+\left(t_1e^{-rt_1}-t_de^{-rt_d}\right)+\frac{e^{-rt_1}-e^{-rt_d}}{r^2}\right\}-\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(e^{-rt_1}-e^{-rt_d}\right)\right]\Bigg\}\Bigg]\left(1-\lambda\left(1-e^{-\sigma G_T}\right)\right)-C_s\left[D\beta t_2\left(\frac{e^{-rt_1}-e^{-rT}}{r}\right)-D\beta\left(e^{-rT}\left(\frac{1}{r^2}-\frac{T}{r}\right)+e^{-rt_1}\left(\frac{t_1}{r}-\frac{1}{r^2}\right)\right)\right]+C_l(1-\beta)D\left(\frac{e^{-rt_1}-e^{-rT}}{r}\right)+pI_eD\left[-\frac{Me^{-rM}}{r}+\frac{e^{-rM}}{r^2}-\frac{1}{r^2}\right]-C_pI_c\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-M)-rM}}{\theta\omega(\xi)+r}-\frac{e^{-rM}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(-\frac{t_1e^{-rt_1}}{\theta\omega(\xi)+r}+\frac{t_1e^{-rt_1}}{r}+\frac{e^{-rt_1}}{r^2}+\frac{t_1e^{\theta\omega(\xi)(t_1-M)-rM}}{\theta\omega(\xi)+r}-\frac{Me^{-rM}}{r}-\frac{e^{-rM}}{r^2}\right)-\frac{\left(e^{-rt_1}-e^{-rM}\right)}{r}\left(\frac{c}{(\theta\omega(\xi))^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\right)\right]$$

$$TP_3(p,t_d,t_1,T)=O_r+C_h\left[\left(Dt_d+\frac{c}{2}t_d^2\right)t_d-\left(\frac{t_de^{-rt_d}}{-r}-\frac{e^{-rt_d}}{r^2}+\frac{1}{r^2}\right)\left(1+\frac{c}{2}\right)-\left(e^{-rt_d}-1\right)\left\{\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\right.\right.$$

$$\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_d}\right)\Big\}\Big] + C_h\left[\frac{D}{\theta\omega(\xi)}\left\{\frac{2\left(e^{-rt_1}-e^{-rt_d}\right)}{r}+\frac{1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-\right.\right.\right.$$
$$\left.\left. e^{-rt_1}\right)\right\}+\frac{c}{\theta\omega(\xi)}\left\{\frac{t_1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-e^{-rt_1}\right)+\left(t_1e^{-rt_1}-t_de^{-rt_d}\right)+\right.$$
$$\left.\frac{e^{-rt_1}-e^{-rt_d}}{r^2}\right\}-\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(e^{-rt_1}-e^{-rt_d}\right)\Big]+C_h'\left[\left(-\frac{t_de^{-rt_d}}{r}-\frac{e^{-rt_d}}{r^2}+\right.\right.$$
$$\left.\frac{1}{r^2}\right)\left(Dt_d+\frac{c}{2}t_d^2+\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\right.$$
$$\left.\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)(t_d)}\right)\right)-D\left(-\frac{t_d^2e^{-rt_d}}{r}-\frac{2t_de^{-rt_d}}{r^2}-\frac{2e^{-rt_d}}{r^3}+\frac{2}{r^3}\right)-\frac{c}{2}\left(-\frac{t_d^3e^{-rt_d}}{r}-\right.$$
$$\left.\left.\frac{3t_d^2e^{-rt_d}}{r^2}+\frac{6t_de^{-rt_d}}{r^4}+6/r^4\right)\right]+C_h'\left[\frac{D}{\theta\omega(\xi)}\left(e^{-rt_1}\left(\frac{t_1}{r}+\frac{1}{r^2}-\frac{1}{(\theta\omega(\xi)+r)^2}-\frac{t_1}{\theta\omega(\xi)+r}\right)+\right.\right.$$
$$\left.e^{-rt_d}\left(-\frac{t_d}{r}-\frac{1}{r^2}\right)+e^{\theta\omega(\xi)(t_1-t_d)-rt_d}\left(\frac{t_d}{\theta\omega(\xi)+r}+\frac{1}{(\theta\omega(\xi)+r)^2}\right)\right)+\frac{c}{\theta\omega(\xi)}\left(e^{-rt_1}\left(\frac{t_1^2}{r}+\right.\right.$$
$$\left.\frac{2t_1}{r^2}-\frac{2}{r^3}-\frac{t_1^2}{\theta\omega(\xi)+r}-\frac{t_1}{(\theta\omega(\xi)+r)^2}\right)+e^{-rt_d}\left(\frac{2}{r^3}-\frac{2t_d}{r^2}-\frac{t_d^2}{r}\right)+$$
$$\left.e^{\theta\omega(\xi)(t_1-t_d)-rt_d}\left(\frac{t_1t_d}{\theta\omega(\xi)+r}+\frac{t_1}{(\theta\omega(\xi)+r)^2}\right)\right)+\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(e^{-rt_1}\left(-\frac{t_1}{r}-\right.\right.$$
$$\left.\left.\left.\frac{1}{r^2}\right)+e^{-rt_d}\left(\frac{t_d}{r}+\frac{1}{r^2}\right)\right)\right]+C_pQ+C_d\omega(\xi)\theta\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\right.\right.$$
$$\left.\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}-\right.$$
$$\left.\left.t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right]+C_{dc}\theta\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\right.\right.$$
$$\left.\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\right.$$
$$\left.\left.\frac{e^{-rt_1}}{r^2}-t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right]+\xi+\left[B+\right.$$
$$HC_{dc}\theta\left[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r}+\frac{e^{-rt_1}}{r}+\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r}-\frac{e^{-rt_d}}{r}\right)+\frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r}-\right.\right.$$
$$\left.t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r}+t_1\frac{e^{-rt_1}}{r}-\frac{e^{-rt_1}}{r^2}-t_d\frac{e^{-rt_d}}{r}+\frac{e^{-rt_d}}{r^2}\right)-\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-\right.$$
$$\left.\left.e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\right]+R\left\{C_h\left[\left(Dt_d+\frac{c}{2}t_d^2\right)t_d-\left(\frac{t_de^{-rt_d}}{-r}-\frac{e^{-rt_d}}{r^2}+\frac{1}{r^2}\right)\left(1+\frac{c}{2}\right)-\right.\right.$$
$$\left(e^{-rt_d}-1\right)\left\{\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right)+\frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)}-t_d\right)+\right.$$
$$\left.\left.\frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_d}\right)\right\}\right]+C_h\left[\frac{D}{\theta\omega(\xi)}\left\{\frac{2\left(e^{-rt_1}-e^{-rt_d}\right)}{r}+\frac{1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d}-\right.\right.\right.$$

$$e^{-rt_1})\Big\} + \frac{c}{\theta\omega(\xi)}\Big\{\frac{t_1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d} - e^{-rt_1}\right) + \left(t_1e^{-rt_1} - t_de^{-rt_d}\right) + \frac{e^{-rt_1}-e^{-rt_d}}{r^2}\Big\} - \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(e^{-rt_1}-e^{-rt_d}\right)\Big]\Big\}\Big]\left(1-\lambda\left(1-e^{-\sigma G_T}\right)\right) - C_s\left[D\beta t_2\left(\frac{e^{-rt_1}-e^{-rT}}{r}\right) - D\beta\left(e^{-rT}\left(\frac{1}{r^2}-\frac{T}{r}\right) + e^{-rt_1}\left(\frac{t_1}{r}-\frac{1}{r^2}\right)\right)\right] + C_l(1-\beta)D\left(\frac{e^{-rt_1}-e^{-rT}}{r}\right) + pI_e\left[D\left(-\frac{t_1e^{-rt_1}}{r} - \frac{e^{-rt_1}}{r^2} + \frac{1}{r^2}\right) - \frac{(M-t_1)D}{r}\left(e^{-rt_1}-1\right)\right]$$

$$TP_4(p,t_d,t_1,T) = O_r + C_h\Big[\left(Dt_d + \frac{c}{2}t_d^2\right)t_d - \left(\frac{t_de^{-rt_d}}{-r} - \frac{e^{-rt_d}}{r^2} + \frac{1}{r^2}\right)\left(1+\frac{c}{2}\right) - \left(e^{-rt_d}-1\right)\Big\{\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right) + \frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)} - t_d\right) + \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_d}\right)\Big\}\Big] + C_h\Big[\frac{D}{\theta\omega(\xi)}\Big\{\frac{2\left(e^{-rt_1}-e^{-rt_d}\right)}{r} + \frac{1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d} - e^{-rt_1}\right)\Big\} + \frac{c}{\theta\omega(\xi)}\Big\{\frac{t_1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d} - e^{-rt_1}\right) + \left(t_1e^{-rt_1} - t_de^{-rt_d}\right) + \frac{e^{-rt_1}-e^{-rt_d}}{r^2}\Big\} - \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(e^{-rt_1}-e^{-rt_d}\right)\Big] + C_h'\Bigg[\left(-\frac{t_de^{-rt_d}}{r} - \frac{e^{-rt_d}}{r^2} + \frac{1}{r^2}\right)\Bigg(Dt_d + \frac{c}{2}t_d^2 + \frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)}-1\right) + \frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)} - t_d\right) + \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)(t_d)}\right)\Bigg) - D\left(-\frac{t_d^2e^{-rt_d}}{r} - \frac{2t_de^{-rt_d}}{r^2} - \frac{2e^{-rt_d}}{r^3} + \frac{2}{r^3}\right) - \frac{c}{2}\left(-\frac{t_d^3e^{-rt_d}}{r} - \frac{3t_d^2e^{-rt_d}}{r^2} + \frac{6t_de^{-rt_d}}{r^4} + 6/r^4\right)\Bigg] + C_h'\Bigg[\frac{D}{\theta\omega(\xi)}\Bigg(e^{-rt_1}\left(\frac{t_1}{r} + \frac{1}{r^2} - \frac{1}{(\theta\omega(\xi)+r)^2} - \frac{t_1}{\theta\omega(\xi)+r}\right) + e^{-rt_d}\left(-\frac{t_d}{r} - \frac{1}{r^2}\right) + e^{\theta\omega(\xi)(t_1-t_d)-rt_d}\left(\frac{t_d}{\theta\omega(\xi)+r} + \frac{1}{(\theta\omega(\xi)+r)^2}\right)\Bigg) + \frac{c}{\theta\omega(\xi)}\Bigg(e^{-rt_1}\left(\frac{t_1^2}{r} + \frac{2t_1}{r^2} - \frac{2}{r^3} - \frac{t_1^2}{\theta\omega(\xi)+r} - \frac{t_1}{(\theta\omega(\xi)+r)^2}\right) + e^{-rt_d}\left(\frac{2}{r^3} - \frac{2t_d}{r^2} - \frac{t_d^2}{r}\right) + e^{\theta\omega(\xi)(t_1-t_d)-rt_d}\left(\frac{t_1t_d}{\theta\omega(\xi)+r} + \frac{t_1}{(\theta\omega(\xi)+r)^2}\right)\Bigg) + \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(e^{-rt_1}\left(-\frac{t_1}{r} - \frac{1}{r^2}\right) + e^{-rt_d}\left(\frac{t_d}{r} + \frac{1}{r^2}\right)\right)\Bigg] + C_pQ + C_d\omega(\xi)\theta\Bigg[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r} + \frac{e^{-rt_1}}{r} + \frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r} - \frac{e^{-rt_d}}{r}\right) + \frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r} - t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r} + t_1\frac{e^{-rt_1}}{r} - \frac{e^{-rt_1}}{r^2} - t_d\frac{e^{-rt_d}}{r} + \frac{e^{-rt_d}}{r^2}\right) - \frac{c}{\left(\theta\omega(\xi)\right)^2}\left(1-e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1}-e^{-rt_d}}{r}\right)\Bigg] + C_{dc}\theta\Bigg[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r} + \frac{e^{-rt_1}}{r} + \frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r} - \frac{e^{-rt_d}}{r}\right) + \frac{c}{\theta\omega(\xi)}\left(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r} - t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r} + t_1\frac{e^{-rt_1}}{r} -\right.$$

$$\frac{e^{-rt_1}}{r^2} - t_d\frac{e^{-rt_d}}{r} + \frac{e^{-rt_d}}{r^2}\Big) - \frac{c}{(\theta\omega(\xi))^2}\left(1 - e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1} - e^{-rt_d}}{r}\right)\Bigg] + \xi + \Bigg[B + HC_{dc}\theta\Bigg[\frac{D}{\theta\omega(\xi)}\left(\frac{e^{-rt_1}}{-\theta\omega(\xi)-r} + \frac{e^{-rt_1}}{r} + \frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{\theta\omega(\xi)+r} - \frac{e^{-rt_d}}{r}\right) + \frac{c}{\theta\omega(\xi)}\Big(\frac{t_1e^{-rt_1}}{-\theta\omega(\xi)-r} - t_1\frac{e^{\theta\omega(\xi)(t_1-t_d)-rt_d}}{-\theta\omega(\xi)-r} + t_1\frac{e^{-rt_1}}{r} - \frac{e^{-rt_1}}{r^2} - t_d\frac{e^{-rt_d}}{r} + \frac{e^{-rt_d}}{r^2}\Big) - \frac{c}{(\theta\omega(\xi))^2}\left(1 - e^{\theta\omega(\xi)t_1}\right)\left(\frac{e^{-rt_1} - e^{-rt_d}}{r}\right)\Bigg] + R\Bigg\{C_h\Bigg[\left(Dt_d + \frac{c}{2}t_d^2\right)t_d - \left(\frac{t_de^{-rt_d}}{-r} - \frac{e^{-rt_d}}{r^2} + \frac{1}{r^2}\right)\left(1 + \frac{c}{2}\right) - (e^{-rt_d} - 1)\Bigg\{\frac{D}{\theta\omega(\xi)}\left(e^{\theta\omega(\xi)(t_1-t_d)} - 1\right) + \frac{c}{\theta\omega(\xi)}\left(t_1e^{\theta\omega(\xi)(t_1-t_d)} - t_d\right) + \frac{c}{(\theta\omega(\xi))^2}\left(1 - e^{\theta\omega(\xi)t_d}\right)\Bigg\}\Bigg] + C_h\Bigg[\frac{D}{\theta\omega(\xi)}\Bigg\{\frac{2\left(e^{-rt_1} - e^{-rt_d}\right)}{r} + \frac{1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d} - e^{-rt_1}\right)\Bigg\} + \frac{c}{\theta\omega(\xi)}\Bigg\{\frac{t_1}{\theta\omega(\xi)+r}\left(e^{\theta\omega(\xi)(t_1-t_d)-rt_d} - e^{-rt_1}\right) + \left(t_1e^{-rt_1} - t_de^{-rt_d}\right) + \frac{e^{-rt_1} - e^{-rt_d}}{r^2}\Bigg\} - \frac{c}{(\theta\omega(\xi))^2}\left(1 - e^{\theta\omega(\xi)t_1}\right)\left(e^{-rt_1} - e^{-rt_d}\right)\Bigg]\Bigg\}\Bigg]\left(1 - \lambda\left(1 - e^{-\sigma G_T}\right)\right) - C_s\left[D\beta t_2\left(\frac{e^{-rt_1} - e^{-rT}}{r}\right) - D\beta\left(e^{-rT}\left(\frac{1}{r^2} - \frac{T}{r}\right) + e^{-rt_1}\left(\frac{t_1}{r} - \frac{1}{r^2}\right)\right)\right] + C_l(1 - \beta)D\left(\frac{e^{-rt_1} - e^{-rT}}{r}\right) + pI_e\left[D\left(-\frac{t_1e^{-rt_1}}{r} - \frac{e^{-rt_1}}{r^2} + \frac{1}{r^2}\right) + \frac{(M-T)D}{r}\left(1 - e^{-rT}\right)\right]$$

Biographical Notes

Anil Kumar is M.Sc. and M.Phil. in Mathematics. He has also been a Research Fellow with the Council of Scientific and Industrial Research, India. He holds a Ph.D. in Operation Research. He has supervised several dissertations for the award of degree in M.Sc. He has several published research papers to his credit. He is currently working as a Professor in the Department of Mathematics at MMH College Ghaziabad U.P. India.

Monu Kumar is M.Sc. and M.Phil. in Mathematics. He has also been a Research Fellow with the Council of Scientific and Industrial Research, India. He has several published research papers to his credit. Currently he is Research Scholar in the Department of Mathematics at MMH College Ghaziabad U.P. India.

S. R. Singh is M.Sc. and M.Phil. in Mathematics. He has also been a Junior Research Fellow with the Council of Scientific and Industrial Research, India. He holds a Ph.D. in Inventory Management and is supervising several research scholars for the degree of Ph.D. He has supervised several dissertations for the award of degree in M.Sc. and M.Phil. He has several published research papers to his credit. He has awarded by Sarswati Saman in 2019 by UP Government. He is currently working as an Professor and Head in the Department of Mathematics at C.C.S.University Meerut U.P, India.

Deepi D is M.Sc. in Mathematics. Currently she is research scholar in Department of Mathematics C.C.S. University Meerut, India.